International Journal of Engineering, Science and Mathematics
Vol. 6 Issue 4, August 2017,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

# Solutions of Pell's Equation Involving Star Primes

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**Abstract**

We illustrate recent development in computational number theory by studying their implications in solving the Pell's equation. In this paper, we search for finding non – trivial integral solutions to the Pell's equation $x^2 = 73y^2 - 37^t$ for all choices of $t \in \boldsymbol{N}$. Recurrence relations among the solutions are also obtained

***Keywords:***

Pell equation,
Integer solutions,
Diophantine equation,
Emirps,
Star Primes,
Pythagorean Primes.

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## 1. Introduction

The Pell's equation is the equation $x^2 = dy^2 + 1$ to be solved in positive integer $x, y$ for a non – zero integer $d$ [1,2,⋯6, 10, 11]. For example, for $d = 5$ one can take $x = 9, y = 4$. We shall always assume that $d$ is positive but not a square, since otherwise there are clearly no solutions. Pell's equation has an extra ordinarily rich history to which Weil [7] is the best guide. A particularly lucid exposition of method of solving the Pell equation is found in Euler's algebra [9].

Star prime is a star number that is prime. Here using two consecutive star primes 37 & 73 we form a Pell's equation $x^2 = 73y^2 - 37^t$, $t \in \boldsymbol{N}$ and search for its non- trivial integer solutions. In addition, 37 &73 are Pythagorean Primes also.

This communication concerns with the Pell equation $x^2 = 73y^2 - 37^t, t \in \boldsymbol{N}$, and infinitely many positive integer solutions are obtained for the choices of $t$ given by $(i)\, t = 1$, $(ii)\, t = 3$ $(iii) t = 5$ $(iv)\, 2k$ and $t = 2k + 5$. A few interesting relations among the solutions are presented. Further recurrence relations on the solutions are derived.

**Proposition 1:[8]**

Let $p$ be a prime. The negative Pell's equation

$$x^2 - py^2 = -1$$

is solvable if and only if $p = 2$ or $p \equiv 1(mod\ 4)$.

This paper concerns with a negative Pell equation

$$x^2 = 73y^2 - 37^t,\ \ t \in \boldsymbol{N}$$

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Here we consider the prime 73 which confirms the existence of integer solutions of using Proposition 1.

## 2. Method of Analysis

### 2.1: Choice 1: $t = 1$

The Pell equation is

$$x^2 = 73y^2 - 37 \tag{1}$$

Let $(x_0, y_0)$ be the initial solution of (1) given by

$$x_0 = 6\ ;\ \ y_0 = 1$$

To find the other solutions of (1), consider the Pell equation

$$x^2 = 73y^2 + 1$$

whose initial solution $(\widetilde{x_n},\ \widetilde{y_n})$ is given by

$$\widetilde{x_n} = \frac{1}{2}\ f_n$$
$$\widetilde{y_n} = \frac{1}{2\sqrt{73}}\ g_n$$

where $f_n = \left(2281249 + 267000\sqrt{73}\right)^{n+1} + \left(2281249 - 267000\sqrt{73}\right)^{n+1}$

$g_n = \left(2281249 + 267000\sqrt{73}\right)^{n+1} - \left(2281249 - 267000\sqrt{73}\right)^{n+1}, \quad n = 0,1,2,\cdots$

Applying Brahma Gupta lemma between $(x_0, y_0)$ and $(\widetilde{x_n},\ \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (1) are obtained as

$$x_{n+1} = \frac{1}{2}\left[6f_n + \sqrt{73}\ g_n\right] \tag{2}$$
$$y_{n+1} = \frac{1}{2\sqrt{73}}\left[\sqrt{73}\ f_n + 6g_n\right] \tag{3}$$

The recurrence relation satisfied by the solutions of (1) are given by

$$x_{n+2} - 534000x_{n+1} + x_n = 0$$
$$y_{n+2} - 534000y_{n+1} + y_n = 0$$

### 2.2 Choices 2: $t = 3$

The Pell equation is

$$x^2 = 73y^2 - 50653 \tag{4}$$

Let $(x_0, y_0)$ be the initial solution of (4) given by

$$x_0 = 1530;\ \ y_0 = 181$$

Applying Brahma Gupta lemma between $(x_0, y_0)$ and $(\widetilde{x_n},\ \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (4) are obtained as

$$x_{n+1} = \frac{1}{2}\left[1530\ f_n + 181\sqrt{73}\ g_n\right] \tag{5}$$
$$y_{n+1} = \frac{1}{2\sqrt{73}}\left[181\sqrt{73}\ f_n + 1530\ g_n\right] \tag{6}$$

The recurrence relations satisfied by the solutions of (4) are given by

$$x_{n+2} - 534000x_{n+1} + x_n = 0$$
$$y_{n+2} - 534000y_{n+1} + y_n = 0$$

### 2.3 Choices 3: $t = 5$

The Pell equation is

$$x^2 = 73y^2 - 69343957 \tag{7}$$

Let $(x_0, y_0)$ be the initial solution of (7) given by

$$x_0 = 325326;\ \ y_0 = 38089$$

Applying Brahma Gupta lemma between $(x_0, y_0)$ and $(\widetilde{x_n},\ \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (7) are obtained as

$$x_{n+1} = \frac{1}{2}\left[325326\ f_n + 38089\sqrt{73}\ g_n\right] \tag{8}$$
$$y_{n+1} = \frac{1}{2\sqrt{73}}\left[38089\sqrt{73}\ f_n + 325326\ g_n\right] \tag{9}$$

The recurrence relations satisfied by the solutions of (7) are given by

$$x_{n+2} - 534000x_{n+1} + x_n = 0$$
$$y_{n+2} - 534000y_{n+1} + y_n = 0$$

**2.4 Choices 4: $t = 2k,\ \ k \in N$**

The Pell equation is

$$x^2 = 73y^2 - 37^{2k},\ k \in \boldsymbol{N} \tag{10}$$

Let $(x_0, y_0)$ be the initial solution of (10) given by

$$x_0 = 37^k.1068\,;\quad y_0 = 37^k.125$$

Applying Brahma Gupta lemma between $(x_0, y_0)$ and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (10) are obtained as

$$x_{n+1} = \frac{37^k}{2}\left[1068 f_n + 125\sqrt{73}\, g_n\right] \tag{11}$$

$$y_{n+1} = \frac{37^k}{2\sqrt{73}}\left[125\sqrt{73}\, f_n + 1068\, g_n\right] \tag{12}$$

The recurrence relations satisfied by the solutions of (10) are given by

$$x_{n+2} - 534000x_{n+1} + x_n = 0$$
$$y_{n+2} - 534000y_{n+1} + y_n = 0$$

**2.5 Choices 5: $t = 2k + 5,\ \ k > 0$**

The Pell equation is

$$x^2 = 17y^2 - 19^{2k+5},\ k > 0 \tag{13}$$

Let $(x_0, y_0)$ be the initial solution of (13) given by

$$x_0 = 37^{k-1}.68826498\,;\quad y_0 = 37^{k-1}.8055613$$

Applying Brahma Gupta lemma between $(x_0, y_0)$ and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solutions to (13) are obtained as

$$x_{n+1} = \frac{37^{k-1}}{2}\left[68826498\, f_n + 8055613\sqrt{73}\, g_n\right] \tag{14}$$

$$y_{n+1} = \frac{37^{k-1}}{2\sqrt{73}}\left[8055613\sqrt{73}\, f_n + 68826498\, g_n\right] \tag{15}$$

The recurrence relations satisfied by the solutions of (13) are given by

$$x_{n+2} - 534000x_{n+1} + x_n = 0$$
$$y_{n+2} - 534000y_{n+1} + y_n = 0$$

## 3. Conclusion

Solving a Pell's equation using the above method provides powerful tool for finding solutions of equations of similar type. Neglecting any time consideration it is possible using current methods to determine the solvability of Pell – like equation.

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